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On a family of differential operators with the coupling parameter in the boundary condition

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Dedicated to Professor Des Evans on the occasion of his 65th birthday

Abstract

We study a family of differential operators $\mathbf{L}_\alpha$ in two variables, depending on the coupling parameter $\alpha \geqslant 0$ that appears only in the boundary conditions. Our main concern is the spectral properties of $\mathbf{L}_\alpha$, which turn out to be quite different for $\alpha < 1$ and for $\alpha > 1$. In particular, $\mathbf{L}_\alpha$ has a unique self-adjoint realization for $\alpha < 1$ and many such realizations for $\alpha > 1$. In the more difficult case $\alpha > 1$ an analysis of non-elliptic pseudodifferential operators in dimension one is involved.

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1. Introduction

In the paper [8] Smilansky suggested a mathematical model which he called 'The irreversible quantum graph'. In this model a one-dimensional quantum graph interacts with a finite system of harmonic oscillators attached at different points of the graph. Regardless of the physical meaning of this model, it is quite interesting from the mathematical point of view, since, being a singular perturbation problem, it exhibits many unusual effects. These effects appear already in the one-oscillator case. They were discussed in the survey paper [9], see also references therein.

In the simplest case (the graph is a real line, with only one oscillator attached) the problem consists in the study of a family of differential operators $\mathbf{A}_\alpha$ on $\mathbb{R}^2$, depending on the coupling parameter $\alpha \geqslant 0$. The differential expression which defines the action of $\mathbf{A}_\alpha$ does not involve α, this parameter appears only in the transmission condition across a straight line in the plane. The operator $\mathbf{A}_0$ admits an exhaustive description via the separation of variables, and the passage to $\mathbf{A}_\alpha$ with $\alpha \neq 0$ can be expressed, at least formally, in the terms of perturbations of quadratic forms. The main peculiarity of the problem stems from the fact that the perturbation is too strong: it is only relatively bounded but not relatively compact with respect to the operator $\mathbf{A}_0$ (in the sense of quadratic forms). For this reason, the standard machinery of the perturbation theory does not work. Still, it turned out to be possible to give a detailed description of the

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spectrum $\sigma(\mathbf{A}_\alpha)$ for all $\alpha > 0$. A borderline value α^* of the parameter α exists, such that the properties of $\sigma(\mathbf{A}_\alpha)$ are quite different for $\alpha < \alpha^*$ and for $\alpha \geqslant \alpha^*$. For $\alpha < \alpha^*$ the absolutely continuous (a.c.) spectrum of $\mathbf{A}_\alpha$ is the same as for $\mathbf{A}_0$, including the multiplicity. Eigenvalues appear below the bottom of $\sigma(\mathbf{A}_0)$, their number grows indefinitely as $\alpha \nearrow \alpha^*$ and satisfies an asymptotic relation of a non-standard type. For $\alpha = \alpha^*$ these eigenvalues disappear and a new branch of the a.c. spectrum appears instead, filling $[0, \infty)$. For α above the threshold α^*, the operator $\mathbf{A}_\alpha$ is not semi-bounded any more and its a.c. spectrum fills the whole real line. Thus, the system exhibits a sort of phase transition as the parameter α crosses the threshold α^*.

The mathematical mechanism behind such a behavior of the spectrum lies in a very special form of the transmission condition for the operator $\mathbf{A}_\alpha$. This condition generates in a natural way an infinite Jacobi matrix which depends on the parameter α and whose spectral properties for $\alpha < \alpha^*$ and for $\alpha \geqslant \alpha^*$ are quite different.

The papers [4,5] are devoted to the case of two oscillators, but actually their results show what happens in the general case of an arbitrary number of oscillators. It was an initiative of Des Evans, to start the work on these papers, and we take pleasure in emphasizing his role in the study of this class of problems.

In the present paper we investigate another family of differential operators, say $\mathbf{L}_\alpha$, of a similar nature. It was also proposed by Smilansky (private communication). Again, all operators in the family are determined by a differential expression not depending on the parameter, and they differ by the transmission condition. Like in the case of the family $\mathbf{A}_\alpha$, a certain family of Jacobi matrices is closely related to the operator. However, the properties of the two families are rather different and another type of phase transition occurs. Namely, for large values of α the operator $\mathbf{L}_\alpha$ has many self-adjoint realizations, and the negative spectrum of each realization is discrete and unbounded from below. The mechanism of this transition lies in an unusual breaking of the Shapiro — Lopatinsky ellipticity condition at several points on the interface line, and the analysis of this situation involves a study of a priori estimates for some non-elliptic pseudodifferential operators.

In the last section of the paper we briefly consider yet another family $\mathbf{M}_\alpha$ of differential operators. It looks rather similar to the family $\mathbf{L}_\alpha$, but some important details in the behavior of the spectrum are quite different.

Taken together, the families $\mathbf{A}_\alpha$, $\mathbf{L}_\alpha$, and $\mathbf{M}_\alpha$ show that presence of the coupling parameter in the boundary condition may cause quite different types of the phase transition. It is tempting to develop a general scheme which would include all these examples as special cases.

2. Stating the problem. Preliminaries

We study a family $\mathbf{L}_\alpha$ of differential operators on the cylinder $\Omega = \mathbb{R} \times \mathbb{S}^1$ identified with the strip $\mathbb{R} \times (0, 2\pi)$ with periodic boundary conditions for all functions involved. Further on, x stands for the co-ordinate on $\mathbb{R}$ and y for the co-ordinate on $\mathbb{S}^1$. The operator $\mathbf{L}_\alpha$ is generated by the Laplacian $-\Delta U = -U''_{x^2} - U''_{y^2}$ and two conditions at $x = 0$. The first condition is the continuity

$$U(0+, y) = U(0-, y)(=U(0, y)) \tag{2.1}$$

and the second one is a ‘transmission condition’ at $x = 0$:

$$U'_x(0+, y) - U'_x(0-, y) = \mathrm{i}\alpha(U'_y(0, y)\cos y + (U(0, y)\cos y)'_y). \tag{2.2}$$

In (2.2) α is a real parameter. The passage $\alpha \mapsto -\alpha$ corresponds to the change of variables $y \mapsto y + \pi$, which does not affect the spectrum. For this reason, it is enough to consider $\alpha \geqslant 0$.

By using the Fourier expansion

$$U = (2\pi)^{-1/2} \sum_{n\in\mathbb{Z}} u_n(x)\mathrm{e}^{\mathrm{i}ny} \tag{2.3}$$

(in short, $U \sim \{u_n\}$), we reduce the problem formally to an infinite system of ordinary differential operators on the real axis,

$$-\Delta U \sim \{-u''_n + n^2 u_n\}, \quad x \neq 0, \qquad n \in \mathbb{Z}, \tag{2.4}$$

coupled by the conditions

$$u_n(0+) = u_n(0-)(=u_n(0)), \tag{2.5}$$

$$u_n'(0+) - u_n'(0-) = -\alpha((n+1/2)u_{n+1}(0) + (n-1/2)u_{n-1}(0)). \tag{2.6}$$

The operator $\mathbf{L}_0$ is just the standard Laplacian on the cylinder Ω, with the domain $H^2(\Omega)$. Thus, for $\alpha = 0$ the above formal reduction of the partial differential operator is legal, the system decouples, and we get

$$\mathbf{L}_0 = \sum_{n\in\mathbb{Z}}^{\oplus} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + n^2\right). \tag{2.7}$$

Here $-\mathrm{d}^2/\mathrm{d}x^2$ stands for the self-adjoint operator in $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$ and the symbol $\sum^{\oplus}$ denotes the orthogonal sum of operators. The expansion (2.7) leads to the complete description of the spectrum $\sigma(\mathbf{L}_0)$: it is absolutely continuous, fills the half-line $[0, \infty)$, and its multiplicity function is given by

$$\mathfrak{m}_{\text{a.c.}}(\lambda; \mathbf{L}_0) = 2 + 4[\lambda] \quad \forall \lambda \geqslant 0, \tag{2.8}$$

where, as usual, $[\lambda]$ denotes the integer part of a real number λ.

For $\alpha \neq 0$ we must first specify in what sense the conditions (2.1) and (2.2) are understood. Suppose that $U \in L^2(\Omega)$ is a weak solution of the equation $-\Delta U = F \in L^2$ in each semi-cylinder

$$\Omega_\pm = \{(x, y) \in \Omega: \pm x > 0\}.$$

Take any $\Lambda \in \mathbb{C}\backslash[0, \infty)$ and consider the function:

$$U_0 = (\mathbf{L}_0 - \Lambda)^{-1}(F - \Lambda U) \in H^2(\Omega). \tag{2.9}$$

The function $W = U - U_0$ belongs to $L^2(\Omega)$ and satisfies the equation $\Delta W + \Lambda W = 0$ in each semi-cylinder $\Omega_\pm$. Let $W_\pm$ stand for the restriction of W to $\Omega_\pm$. The functions $W_\pm$ can be expanded in the Fourier series

$$W_\pm(x, y) = \sum_n w_n^\pm \mathrm{e}^{\mathrm{i}ny} \mathrm{e}^{-|x|\sqrt{n^2-\Lambda}}, \quad \mathrm{Re}\sqrt{n^2 - \Lambda} \geqslant 0. \tag{2.10}$$

Both series converge in $L^2(\Omega_\pm)$ and

$$\int_{\Omega_\pm} |W_\pm(x, y)|^2 \,\mathrm{d}x\,\mathrm{d}y = \sum_n \frac{|w_n^\pm|^2}{2\,\mathrm{Re}\sqrt{n^2 - \Lambda}}.$$

Hence, $W_\pm \in L^2(\Omega_\pm)$ is equivalent to $\sum_n |w_n^\pm|^2 (n^2+1)^{-1/2} < \infty$. It follows that for each $x \in \mathbb{R}$ the series in (2.10) converge in $H^{-1/2}(\mathbb{S}^1)$ and moreover, $W_\pm(x, \cdot)$ are continuous as functions of x with values in $H^{-1/2}(\mathbb{S}^1)$. The same is true for the function U, and this explains the meaning of condition (2.1): namely,

$$U(0+, y) = U(0-, y) \quad \text{as distributions in } H^{-1/2}(\mathbb{S}^1). \tag{2.11}$$

Denote by $\mathscr{M}(\Omega)$ the class of all functions $U \in L^2(\Omega)$ which meet the following conditions:

1. The distributions $\Delta(U \restriction \Omega_\pm)$ are functions in $L^2(\Omega_\pm)$,
2. Condition (2.11) is satisfied.

For any $\Lambda \notin [0, \infty)$ we also set

$$\mathscr{M}_\Lambda(\Omega) = \{W \in \mathscr{M}(\Omega): \Delta W + \Lambda W = 0 \text{ in } \Omega_\pm\}.$$

The Fourier expansion of any function $W \in \mathscr{M}_\Lambda(\Omega)$ has the form:

$$W(x, y) = \sum_n w_n \mathrm{e}^{\mathrm{i}ny} \mathrm{e}^{-|x|\sqrt{n^2-\Lambda}}, \tag{2.12}$$

that is, for the coefficients in (2.10) we have $w_n^+ = w_n^-$ $(=w_n)$ and thus the function $W(x,\cdot)$ is even in x. We also conclude that

$$W \in \mathscr{M}_\Lambda(\Omega) \iff W(0,\cdot) \in H^{-1/2}(\mathbb{S}^1).$$

Let us recall that in the terms of the Fourier coefficients w_n the latter inclusion is equivalent to

$$\sum_n |w_n|^2(n^2+1)^{-1/2} < \infty. \tag{2.13}$$

Differentiation in (2.12) shows that for any $W \in \mathscr{M}_\Lambda(\Omega)$ the derivatives $W_y'(x,\cdot)$, $W_x'(x,\cdot)$ take values in the space $H^{-3/2}(\mathbb{S}^1)$. The first of them, being an even function, is continuous in the topology of this space for all $x \in \mathbb{R}^1$. The second one is continuous in the topology of $H^{-3/2}(\mathbb{S}^1)$ for $x \geqslant 0$ and for $x \leqslant 0$ separately, and its jump across the circle $\{x=0\}$ is well defined as an element in $H^{-3/2}(\mathbb{S}^1)$. The decomposition $U = U_0 + W$, where U_0 is defined by (2.9), shows that the same is true for any $U \in \mathscr{M}(\Omega)$. In particular, this gives the precise meaning to both sides in (2.2) as distributions in $H^{-3/2}(\mathbb{S}^1)$.

Substituting the Fourier expansion (2.12) and its differentiated forms into (2.2), we arrive at system (2.4), (2.5), (2.6) which is equivalent to the initial problem.

The following version of the Green formula is implied by the above argument.

Lemma 2.1. *For any $U \in \mathscr{M}(\Omega)$ and $V \in H^2(\Omega)$ (so that $V(0,\cdot) \in H^{3/2}(\mathbb{S}^1)$) we have*:

$$\left(\int_{\Omega_+} + \int_{\Omega_-}\right)(\Delta U\overline{V} - U\overline{\Delta V})\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathbb{S}^1}(U_x'(0+,y) - U_x'(0-,y))\overline{V(0,y)}\,\mathrm{d}y, \tag{2.14}$$

where the integrals on the left-hand side are understood in the sense of distributions on $\Omega_\pm$ and the integral on the right-hand side is understood in the sense of distributions on $\mathbb{S}^1$.

Denote by $\mathscr{B}$ the differential operator appearing in the condition (2.2):

$$\mathscr{B}u = \mathrm{i}(u_y'\cos y + (u\cos y)_y'). \tag{2.15}$$

The operator $\mathscr{B}$ is symmetric as acting in the space $L^2(\mathbb{S}^1)$. If a function $U \in \mathscr{M}(\Omega)$ satisfies (2.2) for some $\alpha \geqslant 0$ and $V \in H^2(\Omega)$, then the following useful equality holds:

$$\left(\int_{\Omega_+} + \int_{\Omega_-}\right)(\Delta U\overline{V} - U\overline{\Delta V})\,\mathrm{d}x\,\mathrm{d}y = -\alpha\int_{\mathbb{S}^1} U(0,y)\overline{\mathscr{B}V(0,y)}\,\mathrm{d}y. \tag{2.16}$$

This is a direct consequence of Lemma 2.1. Indeed, substituting (2.2) into (2.14) and integrating by parts, we arrive at (2.16).

3. The problem of self-adjointness

In order to study self-adjoint realizations of $\mathbf{L}_\alpha$ for $\alpha > 0$, we first of all introduce two sets, $\mathscr{D}_\alpha$ and $\mathscr{D}_\alpha^\bullet$, on which the operator is well defined. It is convenient to do this in the terms of expansion (2.3).

Definition 3.1. An element $U \sim \{u_n\} \in \mathscr{M}(\Omega)$ lies in $\mathscr{D}_\alpha$ if and only if $u_n \restriction \mathbb{R}_\pm \in H^2(\mathbb{R}_\pm)$ for all $n \in \mathbb{Z}$, conditions (2.5) and (2.6) are satisfied, and

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |-u_n'' + n^2 u_n|^2\,\mathrm{d}x < \infty.$$

An element $U \in \mathscr{D}_\alpha$ belongs to $\mathscr{D}_\alpha^\bullet$, if the number of non-zero terms u_n in the expansion of U is finite.

We denote

$$\mathbf{L}_\alpha = -\Delta\restriction\mathscr{D}_\alpha, \quad \mathbf{L}_\alpha^\bullet = -\Delta\restriction\mathscr{D}_\alpha^\bullet.$$

Lemma 3.2. *The operator* $\mathbf{L}_\alpha^\bullet$ *is symmetric and*

$$\mathbf{L}_\alpha = (\mathbf{L}_\alpha^\bullet)^*.$$

The proof is standard and we skip it.

Theorem 3.3. (1) *For* $0 \leqslant \alpha \leqslant 1$ *the operator* $\mathbf{L}_\alpha$ *is self-adjoint and, hence,* $\mathbf{L}_\alpha^\bullet$ *is essentially self-adjoint.*
(2) *For* $\alpha > 1$ *the operator* $\mathbf{L}_\alpha$ *is non-self-adjoint, and the deficiency indices of* $\mathbf{L}_\alpha^\bullet$ *are* $(2, 2)$.

Proof. We have to check whether the equation

$$\mathbf{L}_\alpha W = (\mathbf{L}_\alpha^\bullet)^* W = \Lambda W \tag{3.1}$$

with $\Lambda \neq \overline{\Lambda}$ has non-zero solutions $W \in \mathscr{D}_\alpha$. If W is such a solution, then $W \in \mathscr{M}_\Lambda(\Omega)$ and, by (2.12), each component in the expansion (2.3) for W can be written as $w_n(x) = w_n \mathrm{e}^{-|x|\sqrt{n^2-\Lambda}}$. The coefficients w_n should satisfy conditions (2.6) that turn into

$$(n + 1/2)w_{n+1} - 2\alpha^{-1} w_n \sqrt{n^2 - \Lambda} + (n - 1/2)w_{n-1} = 0. \tag{3.2}$$

The analysis of system (3.2) is similar to the reasoning in [9, Section 4], and is based upon the classical Birkhoff—Adams theorem, see [3, Theorem 8.36]. The formulation of its leading case, which we need for the study of the operator $\mathbf{L}_\alpha$ with $\alpha \neq 1$, is also reproduced in [9]. This theorem deals with one-sided sequences ($n \in \mathbb{N}$ rather than $n \in \mathbb{Z}$ as in our case), and we have to analyze the behavior of w_n for $n \to +\infty$ and for $n \to -\infty$ separately.

For $n \to +\infty$ we find from the theorem that for $\alpha \neq 1$ Eq. (3.2) has two linearly independent solutions $\{w_n^\pm(+)\}$ such that

$$w_n^\pm(+) = (\lambda_+^\pm)^n n^{-1/2}(1 + \mathrm{O}(n^{-1})), \qquad \lambda_+^\pm = \alpha^{-1} \pm \sqrt{\alpha^{-2} - 1}. \tag{3.3}$$

For $n \to -\infty$ we find in the same way that the system has two linearly independent solutions $\{w_n^\pm(-)\}$ such that

$$w_n^\pm(-) = (\lambda_-^\pm)^n |n|^{-1/2}(1 + \mathrm{O}(|n|^{-1})), \qquad \lambda_-^\pm = -\alpha^{-1} \pm \sqrt{\alpha^{-2} - 1}. \tag{3.4}$$

If $\alpha < 1$, we conclude from the above asymptotic formulas that both for $n > 0$ and for $n < 0$ only one of the basic solutions decays as $|n| \to \infty$. Hence, the space of sequences $\{w_n\}$ satisfying (2.13) (or, equivalently, such that $W \in \mathscr{M}_\Lambda(\Omega)$) is no more than one-dimensional. Suppose that $\{w_n\}$ is such a sequence, and apply the following identity for solutions of recurrence equations of the type

$$Q_{n+1}C_{n+1} + P_n C_n + Q_n C_{n-1} = 0, \quad n \in \mathbb{Z},$$

with Q_n real:

$$\sum_{n=-N}^{N} |C_n|^2 \operatorname{Im} P_n = -Q_{N+1} \operatorname{Im}(C_{N+1}\overline{C_N}) - Q_{-N} \operatorname{Im}(C_{-N-1}\overline{C_{-N}}). \tag{3.5}$$

The proof is straightforward and we skip it; cf. (4.23) and (4.24) in [9].

Applying (3.5) to Eq. (3.2), we obtain

$$2\alpha^{-1} \sum_{n=-N}^{N} |w_n|^2 \operatorname{Im}\sqrt{n^2 - \Lambda} = (N + 1/2) \operatorname{Im}(w_{N+1}\overline{w_N} + w_{-N-1}\overline{w_{-N}}).$$

By (3.3), (3.4) the right-hand side vanishes as $N \to \infty$. Since the number $\operatorname{Im}\sqrt{n^2 - \Lambda}$ is negative for $\operatorname{Im}\Lambda > 0$ and positive for $\operatorname{Im}\Lambda < 0$, we conclude that $w_n = 0$ for all $n \in \mathbb{Z}$. It follows that for $\alpha < 1$ the operator $\mathbf{L}_\alpha$ is self-adjoint.

If $\alpha > 1$, then $|\lambda_\pm| = 1$ and by (3.3), (3.4) any solution $\{w_n\}$ satisfies (2.13). This shows that for $\alpha > 1$ the operator $\mathbf{L}_\alpha$ is non-self-adjoint and the deficiency indices of $\mathbf{L}_\alpha^\bullet$ are $(2, 2)$.

Now, let $\alpha=1$. Then the case (c_1) of the Birkhoff—Adams theorem applies, and Eq. (3.2) has two linearly independent solutions of the form:

$$w_n^{\pm} \sim n^{\pm\sqrt{-\Lambda}}, \quad n \to \infty$$

and similarly for $n \to -\infty$. For any non-real Λ only one of such solutions may satisfy (2.13). Using again the identity (3.5), we conclude that Eq. (3.2) has no non-zero solutions satisfying (2.13). Hence, the operator $\mathbf{L}_1$ is self-adjoint. □

4. Using quadratic forms. Spectrum for $\alpha \leqslant 1$

For small α the simplest way to study the spectrum of the operators $\mathbf{L}_\alpha$ is to use quadratic forms. Our argument here follows the same line as in [9]. However, again, as in Section 2, we have to take into account that the sequence $\{u_n\}$ is two-sided.

Integrating by parts in the expression for $(\mathbf{L}_\alpha U, U)$ over the semi-cylinders $\Omega_\pm$, we find for $U \in \mathscr{D}_\alpha^\bullet$:

$$(\mathbf{L}_\alpha U, U) = \int_\Omega |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{S}^1} U'_x(0-, y)\overline{U(0, y)}\,\mathrm{d}y + \int_{\mathbb{S}^1} U'_x(0+, y)\overline{U(0, y)}\,\mathrm{d}y.$$

Taking into account condition (2.2), we obtain

$$\begin{aligned}(\mathbf{L}_\alpha U, U) - \int_\Omega |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y &= \mathrm{i}\alpha \int_{\mathbb{S}^1} (U'_y(0, y)\cos y + (U(0, y)\cos y)'_y)\overline{U(0, y)}\,\mathrm{d}y \\ &= \mathrm{i}\alpha \int_{\mathbb{S}^1} (U'_y(0, y)\overline{U(0, y)} - U(0, y)\overline{U'_y(0, y)})\cos y\,\mathrm{d}y \\ &= -2\alpha \int_{\mathbb{S}^1} \mathrm{Im}(U'_y(0, y)\overline{U(0, y)})\cos y\,\mathrm{d}y.\end{aligned}$$

In representation (2.3) this turns into

$$\mathbf{l}_\alpha[U] := (\mathbf{L}_\alpha U, U) = \mathbf{l}_0[U] - \alpha\mathbf{b}[U], \tag{4.1}$$

where

$$\mathbf{l}_0[U] = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (|u'_n|^2 + n^2|u_n|^2)\,\mathrm{d}x, \tag{4.2}$$

$$\mathbf{b}[U] = \sum_{n\in\mathbb{Z}} (2n-1)\mathrm{Re}(u_n(0)\overline{u_{n-1}(0)}). \tag{4.3}$$

Completing the set $\mathscr{D}_0$ in the metric $\mathbf{l}_0[U] + \|U\|^2_{L^2(\Omega)}$, we obtain a set which we denote by $\mathfrak{d}$. On $\mathfrak{d}$ the quadratic form $\mathbf{l}_0$ is well defined and closed, and the associated self-adjoint operator in $L^2(\Omega)$ is $\mathbf{L}_0$. Along with $\mathfrak{d}$, we need its subspace of co-dimension one,

$$\mathfrak{d}' = \{U \sim \{u_n\} \in \mathfrak{d} : u_0(0) = 0\}.$$

Lemma 4.1. *For any $U \in \mathfrak{d}'$ the following inequality is satisfied*:

$$|\mathbf{b}[U]| \leqslant \mathbf{l}_0[U] - \|u_0\|^2_{L^2(\mathbb{R})}. \tag{4.4}$$

Proof. Denote by $\mathfrak{d}^+$ (by $\mathfrak{d}^-$) the subspace in $\mathfrak{d}$, formed by the elements $U \sim \{u_n\}$ whose all components with $n \leqslant 0$ (with $n \geqslant 0$) are zeroes. For $U \in \mathfrak{d}^\pm$ we have $\mathbf{b}[U] = \pm\mathbf{b}^\pm[U]$ where

$$\mathbf{b}^\pm[U] = \sum_{n>1} (2n-1)\,\mathrm{Re}(u_{\pm n}(0)\overline{u_{\pm(n-1)}(0)}). \tag{4.5}$$

The estimates for $\mathbf{b}^+[U]$ and for $\mathbf{b}^-[U]$ are identical and we carry them out for the 'plus' sign. We derive from (4.5) that

$$|\mathbf{b}^+[U]| \leqslant \sum_{n>1}(n-1/2)(|u_n(0)|^2+|u_{n-1}(0)|^2) \leqslant \sum_{n\geqslant 1} 2n|u_n(0)|^2.$$

Now to the nth term in the last sum we apply the elementary inequality

$$2\gamma|f(0)|^2 \leqslant \int_{\mathbb{R}}(|f'|^2+\gamma^2|f|^2)\,\mathrm{d}x \quad \forall f \in H^1(\mathbb{R}), \quad \gamma>0,$$

with $\gamma = n$. We obtain

$$|\mathbf{b}^+[U]| \leqslant \sum_{n\geqslant 1}\int_{\mathbb{R}}(|u_n'|^2+n^2|u_n|^2)\,\mathrm{d}x.$$

Together with the similar inequality for $\mathbf{b}^-[U]$, this yields (4.4). $\square$

It is not difficult to show that the factor 1 in front of $\mathbf{l}_0[U]$ on the right-hand side of (4.4) cannot be improved.

With Lemma 4.1 at our disposal, it is easy to characterize the spectral properties of the operator $\mathbf{L}_\alpha$ for $\alpha < 1$.

Theorem 4.2. *Let* $0<\alpha\leqslant 1$. *Then*

(1)

$$\sigma_{\mathrm{ess}}(\mathbf{L}_\alpha) = \sigma(\mathbf{L}_0) = [0,\infty).$$

(2) *The negative spectrum of* $\mathbf{L}_\alpha$ *consists of exactly one non-degenerate eigenvalue*.

(3) *If* $\alpha<1$, *then also*

$$\sigma_{\mathrm{a.c.}}(\mathbf{L}_\alpha) = \sigma_{\mathrm{a.c.}}(\mathbf{L}_0) = [0,\infty), \qquad \mathfrak{m}_{\mathrm{a.c.}}(\mathbf{L}_\alpha) = \mathfrak{m}_{\mathrm{a.c.}}(\mathbf{L}_0)$$

(*cf* (2.8)).

The proof of statements (1) and (3) basically repeats the argument in [9, Section 9], and we skip it. To justify statement (2), we first of all note that by Lemma 4.1, for $\alpha<1$ the quadratic form $\mathbf{l}_\alpha$, restricted to the domain $\mathfrak{d}'$, is positive definite and closed. Since $\dim \mathfrak{d}/\mathfrak{d}' = 1$, the quadratic form $\mathbf{l}_\alpha$, considered on the whole of $\mathfrak{d}$, is bounded from below and also closed. The corresponding self-adjoint operator is $\mathbf{L}_\alpha$. For $\alpha = 1$, the quadratic form $\mathbf{l}_\alpha$ is only closable on $\mathfrak{d}$, and the operator $\mathbf{L}_1$ corresponds to the closure of $\mathbf{l}_1$.

This reasoning shows that for $0<\alpha\leqslant 1$ the number of negative eigenvalues of $\mathbf{L}_\alpha$ is no more than one. In order to show that it is exactly one, it is enough to find an element $U \in \mathfrak{d}$ which is such that $\mathbf{l}_\alpha[U]<0$. To this end, we take $U \sim \{u_n\}$ with only two non-zero components u_0, u_1, then the desired inequality is

$$\int_{\mathbb{R}}(|u_0'|^2+|u_1'|^2+|u_1|^2)\,\mathrm{d}x < \alpha\,\mathrm{Re}(u_1(0)\overline{u_0(0)}).$$

It is satisfied, for instance, if we take $u_1(x) = \mathrm{e}^{-|x|}$ and $u_0(x) = \varepsilon^{-1/2}\mathrm{e}^{-\varepsilon|x|}$, with ε sufficiently small.

Remark 4.3. For $\alpha>1$ the quadratic form $\mathbf{a}_\alpha$ is unbounded from below. We have to show that for any $\alpha>1$ and any $M>0$ there exists an element $U \in \mathfrak{d}$, such that

$$\mathbf{a}_\alpha[U] + M\|U\|^2_{L^2(\Omega)} < 0. \tag{4.6}$$

Choose a number $N \in \mathbb{N}$ and take $U \sim \{u_n\}$, where $u_N(x) = e^{-|x|\sqrt{N^2+M}}$, $u_{N-1}(x) = e^{-|x|\sqrt{(N-1)^2+M}}$ and all the other components u_n in the expansion (2.3) are zeroes. Then

$$\int_{\mathbb{R}} (|u_N'|^2 + (N^2 + M)|u_N|^2)\,dx = 2\sqrt{N^2 + M},$$

$$\int_{\mathbb{R}} (|u_{N-1}'|^2 + ((N-1)^2 + M)|u_N|^2)\,dx = 2\sqrt{(N-1)^2 + M},$$

and

$$\mathbf{a}_\alpha[U] + M\|U\|^2_{L^2(\Omega)} = 2(\sqrt{N^2 + M} + \sqrt{(N-1)^2 + M}) - \alpha(2N - 1).$$

It is clear, that for any $\alpha > 1$ the last expression is negative, provided that N is taken large enough, and we are done.

5. The case $\alpha > 1$. Singular solutions

In order to reach a better understanding of self-adjoint realizations of the operator $\mathbf{L}_\alpha$ for $\alpha > 1$, we describe here the behavior of the singular solutions found in Section 3.

For $\alpha > 1$ the asymptotic expressions for $w_n^\pm(\pm)$ as in (3.3) and (3.4) can be re-written in a simplified form. Indeed, set

$$y(\alpha) = \arccos(\alpha^{-1}),$$

then

$$\lambda_+^+ = -\lambda_-^- = e^{iy(\alpha)}, \quad \lambda_-^+ = -\lambda_+^- = e^{-iy(\alpha)}.$$

Therefore,

$$w_n^\pm(+) = e^{\pm iny(\alpha)} n^{-1/2}(1 + O(n^{-1})), \quad n \to +\infty,$$
$$w_n^\pm(-) = (-1)^n e^{\mp iny(\alpha)} |n|^{-1/2}(1 + O(|n|^{-1})), \quad n \to -\infty. \tag{5.1}$$

By (2.3), (2.12), and (5.1), each L^2-solution of Eq. (3.1) can be represented as

$$W(x, y) = W(x, y; +) + K_0 e^{-|x|\sqrt{-\Lambda}} + W(x, y; -),$$

where K_0 is a constant, $W(x, y; +)$ is a certain linear combination of the functions

$$W^\pm(x, y; +) = \sum_{n>0} w_n^\pm(+) n^{-1/2} e^{iny} e^{-|x|\sqrt{n^2-\Lambda}}$$
$$= \sum_{n>0} e^{in(y \pm y(\alpha))} n^{-1/2} e^{-|x|\sqrt{n^2-\Lambda}}(1 + O(n^{-1})), \tag{5.2}$$

and $W(x, y; -)$ is a linear combination of the functions

$$W^\pm(x, y; -) = \sum_{n<0} w_n^\pm(-)|n|^{-1/2} e^{iny} e^{-|x|\sqrt{n^2-\Lambda}}$$
$$= \sum_{n<0} e^{in(y \pm y(\alpha) - \pi)} |n|^{-1/2} e^{-|x|\sqrt{n^2-\Lambda}}(1 + O(|n|^{-1})). \tag{5.3}$$

Note that $\sqrt{n^2 - \Lambda} = |n| + O(n^{-1})$, and hence

$$e^{-|x|\sqrt{n^2-\Lambda}} = e^{-|x||n|}(1 + |x|O(|n|^{-1})).$$

Denote by $V^{\pm}(x, y; \pm)$ the functions obtained by replacing the factors $\mathrm{e}^{-|x|\sqrt{n^2-\Lambda}}$ by $\mathrm{e}^{-|x||n|}$ in each term of the sums in (5.2) and (5.3) and dropping the terms $\mathrm{O}(|n|^{-1})$. The error is a bounded function rapidly decaying as $|x| \to \infty$. We have

$$V^{\pm}(x, y; +) = \sum_{n>0} n^{-1/2}\mathrm{e}^{-n(|x|-\mathrm{i}(y\pm y(\alpha)))}.$$

The behavior of such sums as $|x| - \mathrm{i}(y \pm y(\alpha)) \to 0$ is well known. Say, it can be easily derived from Eqs. (13.11) in Chapter II of the book [10]. Denote

$$z^+_{\pm} = |x| - \mathrm{i}(y \pm y(\alpha)), \quad z^-_{\pm} = |x| - \mathrm{i}(y \pm y(\alpha) - \pi),$$

then

$$V^{\pm}(x, y; +) = C(z^+_{\pm})^{-1/2} + \mathrm{O}(1), \quad z^+_{\pm} \to 0,$$

with an appropriate choice of the branch of the square root, and some constant C. In the same way,

$$V^{\pm}(x, y; -) = C(z^-_{\pm})^{-1/2} + \mathrm{O}(1), \quad z^-_{\pm} \to 0.$$

The reasoning above gives the following description of singular solutions $W(x, y)$. These solutions depend also on the choice of Λ, but the leading terms of their singularities do not. For this reason we do not reflect dependence on Λ in our notations.

Proposition 5.1. *The singular solutions $W(x, y)$ of Eq.* (3.1) *have singularities at the points* $(0, y_j) \in \Omega$, *where* y_j, $j = 1, 2, 3, 4$, *are the points* $\pm y(\alpha)$ *and* $\pm y(\alpha) + \pi (\mathrm{mod}\, 2\pi)$. *The singularity at each point is of the form*:

$$W(x, y) \sim C_j(|x| - \mathrm{i}(y - y_j))^{-1/2} + \mathrm{O}(1).$$

In order to explain the role of these four singular points, let us check the Shapiro–Lopatinsky criterion for the ellipticity of the boundary-value problem $-\Delta U = F$ under conditions (2.1) and (2.2). In our case this criterion determines the point $y \in \mathbb{S}^1$ as regular if and only if the problem

$$-\phi''(t) + \phi(t) = 0, \ \ t \neq 0, \quad \phi'(0+) - \phi'(0-) = \pm 2\alpha \cos y \phi(0)$$

has only trivial bounded continuous solutions on the line $t \in (-\infty, \infty)$. This requirement is violated exactly at the points $y = y_j$, $j = 1, 2, 3, 4$, where $\alpha|\cos y| = 1$, the solution being $\phi(t) = \mathrm{e}^{-|t|}$. On the other hand, for $\alpha \in [0, 1)$ the Shapiro–Lopatinsky condition is satisfied at all transition points. Therefore, every weak solution of the equation $-\Delta U = \Lambda U$ satisfying (2.2) belongs to H^2 in both half-cylinders $\Omega_{\pm}$, so it is non-singular, which explains the self-adjointness.

6. The case $\alpha > 1$. Spectral properties

For $\alpha > 1$, the main technical difficulty stems from the fact that Definition 3.1 does not describe the class $\mathscr{D}_\alpha$ in the terms of standard function spaces on Ω. For this reason, our argument here is rather lengthy.

Let us fix some self-adjoint extension $\hat{\mathbf{L}}_\alpha$ of the operator $\mathbf{L}^\bullet_\alpha$. The spectral properties discussed in this section do not depend on the choice of the extension.

We start by establishing a formula for the difference of resolvents of the operators $\hat{\mathbf{L}}_\alpha$ and $\mathbf{L}_0$. The method for finding this kind of expressions is widely used and was proposed by Birman in [2]. Let first Λ be a non-real number. It belongs to the resolvent sets of both operators $\hat{\mathbf{L}}_\alpha$ and $\mathbf{L}_0$, and we denote by $\hat{\mathbf{R}}_\alpha$, $\mathbf{R}_0$ the corresponding resolvents.

Take some $F, G \in L^2(\Omega)$, and consider the sesqui-linear form:

$$\mathbf{r}[F, G] = ((\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)F, G) = (\hat{\mathbf{R}}_\alpha F, G) - (F, \mathbf{R}_0^* G). \tag{6.1}$$

Denote

$$\hat{\mathbf{R}}_\alpha F = U, \quad \mathbf{R}_0^* G = V,$$

then $U \in \mathscr{D}_\alpha$ and $V \in H^2(\Omega)$. Thus, the quadratic form (6.1) can be re-written as

$$(U, (\mathbf{L}_0 - \overline{\Lambda})V) - ((\hat{\mathbf{L}}_\alpha - \Lambda)U, V) = \left(\int_{\Omega_+} + \int_{\Omega_-}\right)(\Delta U \overline{V} - U\overline{\Delta V})\,\mathrm{d}x\,\mathrm{d}y.$$

Applying (2.16), we arrive at

$$\mathbf{r}[F, G] = \alpha \int_{\mathbb{S}^1} U(0, y)\overline{\mathscr{B}V(0, y)}\,\mathrm{d}y,$$

where $\mathscr{B}$ is the operator (2.15). Hence, the latter equality gives the representation of the operator $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ as

$$\hat{\mathbf{R}}_\alpha - \mathbf{R}_0 = 2\alpha S^* T, \quad T = \Gamma\hat{\mathbf{R}}_\alpha, \quad S = \mathscr{B}\Gamma\mathbf{R}_0^*, \tag{6.2}$$

where Γ stands for the operator of restriction of functions on Ω to the circle $x = 0$. The operator T is bounded from $L^2(\Omega)$ to $H^{-1/2}(\mathbb{S}^1)$, and S is bounded from $L^2(\Omega)$ to $H^{1/2}(\mathbb{S}^1)$, so that S^* is bounded from $H^{-1/2}(\mathbb{S}^1)$ to $L^2(\Omega)$.

Our next step is to derive a pseudodifferential equation for the distribution $w = \Gamma W$, where

$$W = U - V_1 := \hat{\mathbf{R}}_\alpha F - \mathbf{R}_0 F, \quad F \in L^2(\Omega). \tag{6.3}$$

Evidently, $W \in \mathscr{M}_\Lambda(\Omega)$ and thus $w \in H^{-1/2}(\mathbb{S}^1)$. Below we denote by $\mathscr{A}$ the operator $-\mathrm{d}^2/\mathrm{d}y^2$ in $L^2(\mathbb{S}^1)$, extended to distributions on $\mathbb{S}^1$. It follows from representation (2.12) that

$$W'_x(0+, y) - W'_x(0-, y) = -2\sum_n w_n\sqrt{n^2 - \Lambda}\,\mathrm{e}^{\mathrm{i}ny} = -2(\mathscr{A} - \Lambda)^{1/2}w(y).$$

Now, taking into account the transmission conditions for U and for V_1, we find that

$$W'_x(0+, y) - W'_x(0-, y) - \alpha\mathscr{B}W(0, y) = \alpha\mathscr{B}V_1(0, y),$$

or

$$(2(\mathscr{A} - \Lambda)^{1/2} + \alpha\mathscr{B})w = -\alpha\mathscr{B}\Gamma\mathbf{R}_0 F \in H^{1/2}(\mathbb{S}^1). \tag{6.4}$$

The operator $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ is, of course, bounded. We are going to show that, actually, it is compact. The proof is based upon the fact that the operator T in (6.2) acts from $L^2(\Omega)$ not only into $H^{-1/2}(\mathbb{S}^1)$ but into a smaller space, $H^{-\varepsilon}(\mathbb{S}^1)$, for any $\varepsilon > 0$. To show this, we need an a priori estimate for Eq. (6.4). This equation is elliptic for $\alpha < 1$, but for $\alpha > 1$, which is the case we are dealing with, it is degenerate, so some more effort is needed.

Lemma 6.1. *For any $\varepsilon > 0$ there exist constants C, C' such that for any $w \in H^{-1/2}(\mathbb{S}^1)$*

$$\|w\|_{H^{-\varepsilon}(\mathbb{S}^1)} \leqslant C\|2(\mathscr{A} - \Lambda)^{1/2}w + \alpha\mathscr{B}w\|_{H^{1/2}(\mathbb{S}^1)} + C'\|w\|_{H^{-1/2}(\mathbb{S}^1)}, \tag{6.5}$$

provided that the first term on the right-hand side of (6.5) *is finite*.

Proof. Denote by $P_\pm$ the Riesz projections,

$$P_+ f = \pi^{-1}\sum_{k \geqslant 0}(f, \mathrm{e}^{\mathrm{i}ky})\mathrm{e}^{\mathrm{i}ky}, \quad P_- f = \pi^{-1}\sum_{k<0}(f, \mathrm{e}^{\mathrm{i}ky})\mathrm{e}^{\mathrm{i}ky}.$$

Here the sums are understood in the sense of distributions; in particular, if $f \in H^s(\mathbb{S}^1)$, $s \in \mathbb{R}$, both series converge in $H^s(\mathbb{S}^1)$.

The operators $P_\pm$ differ by smoothing operators from pseudodifferential operators on the circle with symbols

$$p_+(y, \eta) = \begin{cases} 1 & \text{if } \eta > 0, \\ 0 & \text{if } \eta < 0; \end{cases} \qquad p_-(y, \eta) = 1 - p_+(y, \eta),$$

see the discussion in [1] about the Fourier series representation of pseudodifferential operators on the circle.

For $w \in H^s(\mathbb{S}^1)$ we denote by $w_\pm$ the distributions $w_\pm = P_\pm w$. The operator $(\mathscr{A} - \Lambda)^{1/2}$ is, up to a smoothing term, the pseudodifferential operator with symbol $(\eta^2 - \Lambda)^{1/2} = |\eta| + \mathrm{O}(|\eta|^{-1})$. As it follows from the composition formulas for pseudodifferential operators in dimension one, the operators in (6.5) commute or almost commute with $P_\pm$:

$$(\mathscr{A} - \Lambda)^{1/2} P_\pm = P_\pm (\mathscr{A} - \Lambda)^{1/2}, \quad \mathscr{B} P_\pm = P_\pm \mathscr{B} + K,$$

with K being a smoothing operator. Thus, up to en error being an operator of order -1, the operator $(\mathscr{A} - \Lambda)^{1/2}$ acts on the components $w_\pm$ as the differentiation, with proper signs:

$$\|(\mathscr{A} - \Lambda)^{1/2} w_\pm \mp \mathrm{i} w'_\pm\|_{H^{1/2}(\mathbb{S}^1)} \leqslant C \|w_\pm\|_{H^{-1/2}(\mathbb{S}^1)}.$$

Therefore, (6.5) will follow as soon as we prove that

$$\|w_\pm\|_{H^{-\varepsilon}(\mathbb{S}^1)} \leqslant C\| \pm w'_\pm + \frac{\alpha}{2\mathrm{i}} \mathscr{B} w_\pm\|_{H^{1/2}(\mathbb{S}^1)} + C'\|w_\pm\|_{H^{-1/2}(\mathbb{S}^1)}. \tag{6.6}$$

Estimate (6.6), even with $-\varepsilon$ replaced by $\frac{3}{2}$ on the left-hand side, would follow automatically, if the operators $\pm 2\mathrm{i}\partial_y + \alpha\mathscr{B}$ were elliptic for both signs $\pm$. This is the case for $|\alpha| < 1$. However, for $|\alpha| \geqslant 1$ these operators have points of degeneracy of ellipticity, i.e., the points where the principal symbols $(\pm 1 + \alpha \cos y)\eta$ vanish. Note that these are exactly the points where the singularities of the singular solutions are located, see Section 5. For such degenerate operators considering the principal symbol is not sufficient for getting a priori estimates, so the influence of lower order terms in $\mathscr{B}$ must be taken into account.

We concentrate on the case of the 'minus' sign in (6.6). Let us denote $h(y) = \alpha \cos y$ and set

$$u = -w'_- + \frac{\alpha}{2\mathrm{i}} \mathscr{B} w_- = (h(y) - 1) w'_- + \tfrac{1}{2} h'(y) w_-.$$

We also set $g = (h(y) - 1)^{1/2} w_-$, with a properly chosen branch of the square root. Note that $g' = (h(y) - 1)^{-1/2} u$. Our next task is to derive an estimate of g in the terms of u, assuming that $u \in H^{1/2}(\mathbb{S}^1)$.

The latter assumption on u implies that the function $(h(y) - 1)^{-1/2} u$ belongs to the space $H^{-\delta}(\mathbb{S}^1)$ for an arbitrarily small $\delta > 0$, say $\delta < \frac{1}{2}$. To justify the above statement, we must show that

$$\left| \int_{\mathbb{S}^1} (h(y) - 1)^{-1/2} u(y) \zeta(y)\, \mathrm{d}y \right| \leqslant C \|u\|_{H^{1/2}(\mathbb{S}^1)} \|\zeta\|_{H^\delta(\mathbb{S}^1)}, \quad \forall \zeta \in H^\delta(\mathbb{S}^1). \tag{6.7}$$

But this follows from the Hölder inequality, since $|h - 1|^{-1/2} \in L^r(\mathbb{S}^1)$ for any $r < 2$, and by the embedding theorem $u \in L^q(\mathbb{S}^1)$ for any $q < \infty$ and $\zeta \in L^{2/(1-2\delta)}(\mathbb{S}^1)$.

It follows from (6.7) that

$$\|g'\|_{H^{-\delta}(\mathbb{S}^1)} = \|(h(y) - 1)^{-1/2} u\|_{H^{-\delta}(\mathbb{S}^1)} \leqslant C \|u\|_{H^{1/2}(\mathbb{S}^1)}.$$

Therefore, the function $g = (h(y) - 1)^{1/2} w_-$ lies in $H^{1-\delta}(\mathbb{S}^1)$ and satisfies the estimate

$$\|g\|_{H^{1-\delta}(\mathbb{S}^1)} \leqslant C \|u\|_{H^{1/2}(\mathbb{S}^1)} + C' \|g\|_{H^{-N}(\mathbb{S}^1)},$$

with N being arbitrarily large.

By the definition of g, we have $w_- = (h(y) - 1)^{-1/2} g$. An estimate, similar to (6.7) (even a simpler one, since $g \in L^\infty(\mathbb{S}^1)$), shows that w_- belongs to $H^{-\varepsilon}(\mathbb{S}^1)$, with the required estimate. □

The estimate, just proved, enables us to establish the compactness of the difference of resolvents $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ and of several related operators and to prove spectral estimates.

Proposition 6.2. *The operator $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ is compact, moreover for its singular numbers $s_n(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)$ the estimate*

$$s_n(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0) = \mathrm{O}(n^{-1/2+\varepsilon}) \tag{6.8}$$

holds for any $\varepsilon > 0$. *Further on*,

$$s_n((\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)\mathbf{R}_0) = \mathrm{O}(n^{-5/2+\varepsilon}), \quad s_n(\mathbf{R}_0(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)) = \mathrm{O}(n^{-5/2+\varepsilon}). \tag{6.9}$$

Proof. It follows from factorization (6.2) that

$$s_n(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0) \leqslant C s_n(T),$$

where we have to consider the operator T as acting from $L^2(\Omega)$ to $H^{-1/2}(\mathbb{S}^1)$. For $F \in L^2(\Omega)$ we define the function W as in (6.3) and take $w = W(0, \cdot)$, then

$$TF = \Gamma\hat{\mathbf{R}}_\alpha F = w + \Gamma\mathbf{R}_0 F.$$

The operator $\Gamma\mathbf{R}_0$ acts from $L^2(\Omega)$ to $H^{3/2}(\mathbb{S}^1)$, and the distribution w satisfies Eq. (6.4), whose right-hand side belongs to $H^{1/2}(\mathbb{S}^1)$. Lemma 6.1 applies and gives $w \in H^{-\varepsilon}(\mathbb{S}^1)$. It follows that the operator T is bounded as acting from $L^2(\Omega)$ to $H^{-\varepsilon}(\mathbb{S}^1)$, and therefore, the singular numbers of the operator $T: L^2(\Omega) \to H^{-1/2}(\mathbb{S}^1)$ are controlled by those of the embedding $H^{-\varepsilon}(\mathbb{S}^1) \to H^{-1/2}(\mathbb{S}^1)$. The latter are of the order $\mathrm{O}(n^{-1/2+\varepsilon})$, whence the required estimate (6.8).

Further on, we factorize the operator $\mathbf{R}_0(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)$ as

$$\mathbf{R}_0(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0) = 2\alpha\mathbf{R}_0 S^* T.$$

Since we already know the singular numbers estimate for the operator $T: L^2(\Omega) \to H^{-1/2}(\mathbb{S}^1)$, it is sufficient for us to consider the operator $\mathbf{R}_0 S^*$ as acting between the spaces $H^{-1/2}(\mathbb{S}^1)$ and $L^2(\Omega)$. It is more convenient to deal with the adjoint operator

$$S\mathbf{R}_0^* = \mathscr{B}\Gamma(\mathbf{R}_0^*)^2 : L^2(\Omega) \to H^{1/2}(\mathbb{S}^1).$$

This operator is bounded as acting from $L^2(\Omega)$ to $H^{5/2}(\mathbb{S}^1)$. Hence, the singular numbers of the same operator but considered as acting between the spaces $L^2(\Omega)$ and $H^{1/2}(\mathbb{S}^1)$ are controlled by those of the embedding operator $H^{5/2}(\mathbb{S}^1) \to H^{1/2}(\mathbb{S}^1)$. The latter are of the order $\mathrm{O}(n^{-2})$. This, together with the estimate for T, proves the second estimate in (6.9). The first estimate in (6.9) follows from the second one by passing to adjoint operators. □

Now we arrive at our main result on the spectrum of the operator $\hat{\mathbf{L}}_\alpha$, $\alpha > 1$.

Theorem 6.3. *For* $\alpha > 1$ *the spectrum of the operator* $\hat{\mathbf{L}}_\alpha$ *consists of the essential spectrum filling the semi-axis* $\lambda \geqslant 0$ *and the eigenvalues below the point* 0. *The set of eigenvalues below the essential spectrum is unbounded from below, may have only* 0 *and* $-\infty$ *as limit points, and for the counting function* $n(t) = \#\{\lambda \in \sigma_{\mathrm{disc}}(\hat{\mathbf{L}}_\alpha), \lambda \in (-t, -t_0)\}$, *with any fixed* $t_0 > 0$, *the estimate holds*

$$n(t) = \mathrm{O}(t^{2+\varepsilon_1}) \quad \text{for any } \varepsilon_1 > 0. \tag{6.10}$$

The absolutely continuous spectrum of $\hat{\mathbf{L}}_\alpha$ *fills the half-line* $\lambda \geqslant 0$ *and its multiplicity function coincides with that of* $\mathbf{L}_0$.

Remark 6.4. Estimate (6.10) is rather rough. The authors believe that a more detailed analysis, based upon a further study of the degenerate Eq. (6.4), would show that the counting function has the asymptotics $n(t) \sim Ct^{1/2}$ as $t \to \infty$. Moreover, we think that the negative eigenvalues do not have 0 as their limit point.

Proof. First, we note that due to Weyl theorem, the essential spectrum of the operators $\hat{\mathbf{R}}_\alpha$ and $\mathbf{R}_0$ is the same, therefore, the essential spectrum of $\hat{\mathbf{L}}_\alpha$ coincides with that of $\mathbf{L}_0$, so it is the half-line $[0, \infty)$. Thus, the spectrum of $\hat{\mathbf{L}}_\alpha$ below 0 may only consist of eigenvalues with possible accumulation points only at 0 and $-\infty$. The latter point must be an accumulation point for eigenvalues since the operator $\hat{\mathbf{L}}_\alpha$ is not semi-bounded from below, see Remark 4.3. The discreteness of the negative spectrum implies that there are real regular points of the operator $\hat{\mathbf{L}}_\alpha$, these are all points below 0, which are not eigenvalues. We fix such regular $\Lambda < 0$ and consider the resolvents $\hat{\mathbf{R}}_\alpha$, $\mathbf{R}_0$ at this point.

Then the above construction of the operator $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ and the estimate for its eigenvalues can be repeated, this time for the chosen real Λ. The spectrum of $\mathbf{R}_0$ coincides with the interval $[0, -\Lambda^{-1}]$, and

$$\hat{\mathbf{R}}_\alpha = \mathbf{R}_0 + (\hat{\mathbf{R}}_\alpha - \mathbf{R}_0).$$

The operator $\mathbf{R}_0$ is non-negative, therefore, for any $\mu < 0$ the number of eigenvalues of $\hat{\mathbf{R}}_\alpha$ (counting multiplicities) in $(-\infty, \mu)$ is not greater than the number of eigenvalues of $\hat{\mathbf{R}}_\alpha - \mathbf{R}_0$ in the same interval. The latter quantity is estimated by means of the eigenvalue bound (6.8), which under an appropriate choice of $\varepsilon = \varepsilon(\varepsilon_1)$ leads to (6.10), with $t_0 = -\Lambda$ and $t = -(\mu^{-1} + \Lambda)$.

In order to justify the statement on the absolute continuous spectrum, let us consider the difference $\hat{\mathbf{R}}_\alpha^3 - \mathbf{R}_0^3$. We have

$$\hat{\mathbf{R}}_\alpha^3 - \mathbf{R}_0^3 = (\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)^3 + \hat{\mathbf{R}}_\alpha \mathbf{R}_0 (\hat{\mathbf{R}}_\alpha - \mathbf{R}_0) + \hat{\mathbf{R}}_\alpha (\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)\mathbf{R}_0 + \mathbf{R}_0(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0)^2 + \mathbf{R}_0^2(\hat{\mathbf{R}}_\alpha - \mathbf{R}_0),$$

and due to estimates (6.8) and (6.9) each term is trace class. By Kato's theorem, the absolute continuous parts of operators $\hat{\mathbf{L}}_\alpha$ and $\mathbf{L}_0$ are unitary equivalent. □

7. An alternative model

Here we briefly describe an alternative model, where a slight change in the setting leads to some major changes in the spectral behavior. The family $\mathbf{M}_\alpha$ of differential operators acts on the strip $\Omega' = \mathbb{R} \times (0, \pi)$ and is generated by the Laplacian $-\Delta U = -U''_{x^2} - U''_{y^2}$, the Dirichlet condition $U(x, 0) = U(x, \pi) = 0$, and two additional conditions at $x = 0$:

$$U(0+, y) = U(0-, y) \ (= U(0, y)),$$

$$U'_x(0+, y) - U'_x(0-, y) = -\mathrm{i}\alpha(U'_y(0, y)\sin y + (U(0, y)\sin y)'_y),$$

cf. (2.1), (2.2).

The Fourier expansion for this case has the form:

$$U = \sum_{n=1}^{\infty} u_n(x)\varphi_n(y), \quad \varphi_n(y) = \sqrt{\frac{2}{\pi}}\sin ny$$

(in short, $U \sim \{u_n\}$). The equation and the boundary and transmission conditions reduce to an infinite system of ordinary differential operators on the real axis, coupled by the conditions at $x = 0$:

$$-\Delta U \sim \{-u''_n + n^2 u_n\}, \quad n \in \mathbb{N},$$

each u_n is continuous at $x = 0$;

$$u'_n(0+) - u'_n(0-) = \mathrm{i}\alpha((n + 1/2)u_{n+1}(0) - (n - 1/2)u_{n-1}(0)),$$

with u_0 taken to be identically zero.

For $\alpha = 0$ the system decouples, and we get an analogue of (2.7), but this time with summation over $n \in \mathbb{N}$. From here we derive that the spectrum $\sigma(\mathbf{M}_0)$ is absolutely continuous, fills the half-line $[1, \infty)$, and its multiplicity function is given by

$$\mathfrak{m}_{\text{a.c.}}(\lambda; \mathbf{M}_0) = 2[\lambda] \quad \forall \lambda \geqslant 1. \tag{7.1}$$

It is these two differences with $\mathbf{L}_0$, the sequence of u_n being one-sided and the spectrum of the unperturbed problem starting at 1 rather than at 0, that lead to the changes in the spectral properties of the perturbed operator.

The study of the self-adjointness of $\mathbf{M}_\alpha$ for $\alpha > 0$ follows the same line as for the operators $\mathbf{L}_\alpha$ in Section 3. It turns out that the operator $\mathbf{M}_\alpha$, considered on the natural domain (cf. Definition 3.1), is self-adjoint for $\alpha \leqslant 1$. If $\alpha > 1$, the operator has a one-parameter family $\hat{\mathbf{M}}_\alpha$ of self-adjoint realizations. The singular solutions, which define these realizations by v. Neumann's scheme, have two singular points $(0, y^\pm)$, with singularities of the order $C(|x| + \mathrm{i}(y - y^\pm))^{-1/2}$. The points $y^\pm$ are the solutions of the equation $\alpha \sin y = 1$, these are exactly the points where the Shapiro–Lopatinsky condition is violated.

Similarly to the cylinder case, the spectral analysis of the operator $\mathbf{M}_\alpha$ for $0 < \alpha < 1$ is based upon considering the quadratic forms. The quadratic form for $\mathbf{M}_\alpha$ is

$$\mathbf{m}_\alpha[U] := (\mathbf{M}_\alpha U, U) = \mathbf{m}_0[U] - \alpha\mathbf{b}[U],$$

where

$$\mathbf{m}_0[U] = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} (|u_n'|^2 + n^2|u_n|^2)\,\mathrm{d}x,$$

$$\mathbf{b}[U] = \sum_{n\geqslant 2} (2n-1)\mathrm{Im}(u_n(0)\overline{u_{n-1}(0)}),$$

cf. (4.1)–(4.3). The quadratic form $\mathbf{m}_0$ is positive definite and closed on its natural domain which we again denote by $\mathfrak{d}$. The associated self-adjoint operator in $L^2(\Omega')$ is $\mathbf{M}_0$. The inequality

$$|\mathbf{b}[U]| \leqslant \mathbf{m}_0[U], \quad U \in \mathfrak{d}, \tag{7.2}$$

is checked in the same way as (4.4), and this time no second term as in (4.4) appears. The constant factor 1 in estimate (7.2) is sharp. Hence, for $\alpha < 1$ the quadratic form $\mathbf{m}_\alpha$ is positive definite and closed on $\mathfrak{d}$. The corresponding self-adjoint operator in $L^2(\Omega')$ is $\mathbf{M}_\alpha$. It is not difficult to show that for $\alpha > 1$ the quadratic form $\mathbf{m}_\alpha$ is unbounded from below.

We pass now to the description of the spectrum of $\mathbf{M}_\alpha$. It is here where the differences with $\mathbf{L}_\alpha$ manifest themselves, cf. Theorem 4.2.

Theorem 7.1. *Let* $0 < \alpha < 1$. *Then*

(1)

$$\sigma_{\mathrm{ess}}(\mathbf{M}_\alpha) = \sigma(\mathbf{M}_0) = [1, \infty).$$

(2)

$$\sigma_{\mathrm{a.c.}}(\mathbf{M}_\alpha) = \sigma_{\mathrm{a.c.}}(\mathbf{M}_0) = [1, \infty), \quad \mathfrak{m}_{\mathrm{a.c.}}(\mathbf{M}_\alpha) = \mathfrak{m}_{\mathrm{a.c.}}(\mathbf{M}_0)$$

(*cf.* (7.1)).

(3) *The spectrum of* $\mathbf{M}_\alpha$ *below the threshold* $\lambda_0 = 1$ *is finite*.

We skip the proof which basically repeats the argument in [9, Section 9]. Note that one can also prove that for the pairs $\mathbf{M}_\alpha, \mathbf{M}_0$ and $\mathbf{M}_0, \mathbf{M}_\alpha$ there exist complete isometric wave operators.

The quadratic form $\mathbf{m}_1 \restriction \mathfrak{d}$ is non-negative and closable, it generates the operator $\mathbf{M}_1$. It is possible to show that its essential spectrum is the half-line $[0, \infty)$.

The analysis of the discrete spectrum of $\mathbf{M}_\alpha$ for $\alpha \in (0, 1)$ is based upon a version of Birman–Schwinger principle found in [9]. Before giving its formulation, let us recall the following well-known notations. Given a real number λ and self-adjoint operator Q, whose spectrum on $(-\infty, \lambda)$ is discrete, we write $N_-(\lambda; Q)$ for the number of the eigenvalues $\lambda_n(Q) < \lambda$, counted according to their multiplicities. We also write $N_+(\lambda; Q) = N_-(-\lambda; -Q)$.

It turns out that within an error which is no greater than 1, the number $N_-(1; \mathbf{M}_\alpha)$ coincides with $N_+(\alpha^{-1}; \mathbf{J})$, where $\mathbf{J}$ is a certain infinite Jacobi matrix:

$$0 \leqslant N_-(1; \mathbf{M}_\alpha) - N_+(\alpha^{-1}; \mathbf{J}) \leqslant 1. \tag{7.3}$$

The reasoning is the same as in [9], however, the Jacobi matrix $\mathbf{J}$ turns out to be different: it is the zero-diagonal Jacobi matrix, with the non-diagonal entries given by

$$2j_{n,n-1} = 2j_{n-1,n} = \frac{n-1/2}{(n^2-1)^{1/4}(n^2-2n)^{1/4}}.$$

Since $j_{n,n-1} \to \frac{1}{2}$ as $n \to \infty$, the matrix $\mathbf{J}$ has the absolutely continuous spectrum filling the segment $[-1, 1]$ and the spectrum outside this segment is discrete. Note that $\alpha \in (0, 1)$ is equivalent to $\alpha^{-1} > 1$, so that both terms in (7.3) are finite.

In order to estimate $N_+(\mu; \mathbf{J})$, $\mu = \alpha^{-1}$, we use the asymptotics of $j_{n,n-1}$:

$$j_{n,n-1} \sim \frac{1}{2} + \frac{1}{2}n^{-2} + o(n^{-2}), \quad n \to \infty. \tag{7.4}$$

Using the results of Geronimo [6,7], combined with some standard variational tools, one can show that $N_+(\mu; \mathbf{J})$ can be estimated from below and from above by $|\log(\mu - 1)|$, with different constants. Thus, the number of eigenvalues of $\mathbf{M}_\alpha$ in (0, 1) grows logarithmically as $\alpha \nearrow 1$. We believe that actually a logarithmical asymptotics for the eigenvalues holds.

When α becomes larger than 1, the phase transition occurs, similar to the cylinder case. Each self-adjoint realization $\hat{\mathbf{M}}_\alpha$ of the operator $\mathbf{M}_\alpha$ is unbounded from below, with the spectrum below the point 1 being discrete. The absolutely continuous spectrum is still the half-line $[1, \infty)$, with the same multiplicity function as for $\mathbf{M}_0$. All these properties are proved using the methods exposed in Section 6. Some additional technical complications are caused by the fact that now we should prove estimates of type (6.5) for the operators on an interval $(0, \pi)$, rather than on the circle $\mathbb{S}^1$ which is a manifold without boundary. But these complications can be overcome.

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